Data Analysis, Statistics, Machine Learning

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Predicting

Most statistical prediction models take one of two forms

$y = \Sigma_j(\beta_j x_j) + \varepsilon$ (additive function)

$y = f(x_j, \varepsilon)$ (nonlinear function)

The distinction is important

- The first form is called an additive model
- The second form is called a nonlinear model
- Additive models can be curvilinear (if terms are nonlinear)
- Nonlinear models cannot be transformed to linear

Examples of linear or linearizable models are

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p + \varepsilon$$

$$y = \alpha e^{\beta x + \varepsilon}$$

Examples of nonlinear models are

$$y = \beta_1 x_1 / \beta_2 x_2 + \varepsilon$$

$$y = \log \beta_1 x_1 \varepsilon$$

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Predicting

Regression predicts a set of values on a variable *y* from values on one or more *x* variables.

This is done by fitting a mathematical function that, for any value(s) on the *x* variable(s), yields the most probable value of *y*.

Simple linear model is

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Estimates are

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

The badness or loss of this prediction in a sample of values is represented by the discrepancies between the *y* values and their corresponding predicted values.

$$\text{loss} = \sum_{i=1}^{n} (y - \hat{y})^2$$

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Predicting

Ordinary Least Squares (OLS)

Legendre (1805)

> "Of all the principles that can be proposed for this purpose, I think there is none more general, more exact, or easier to apply, than that which we have used in this work; it consists of making the sum of the squares of the errors a *minimum*. By this method, a kind of equilibrium is established among the errors which, since it prevents the extremes from dominating, is appropriate for revealing the state of the system which most nearly approaches the truth." (translation by Stigler, 1986)

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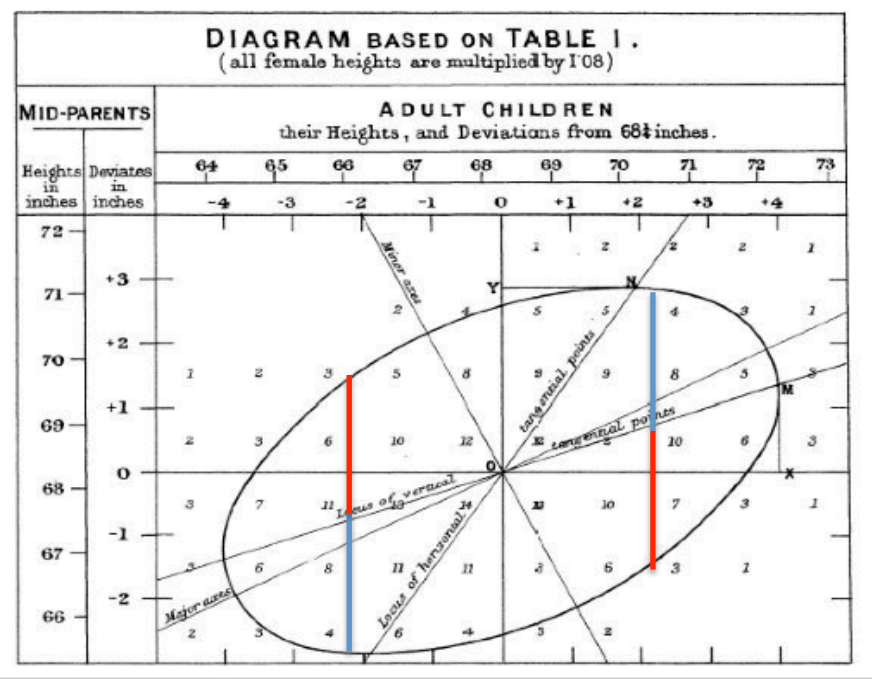
Predicting

Regression

Francis Galton (1887)

He noticed that the blue lines were the same length as the red ones

That is, the line best predicting Y from X "regressed" away from the major axis



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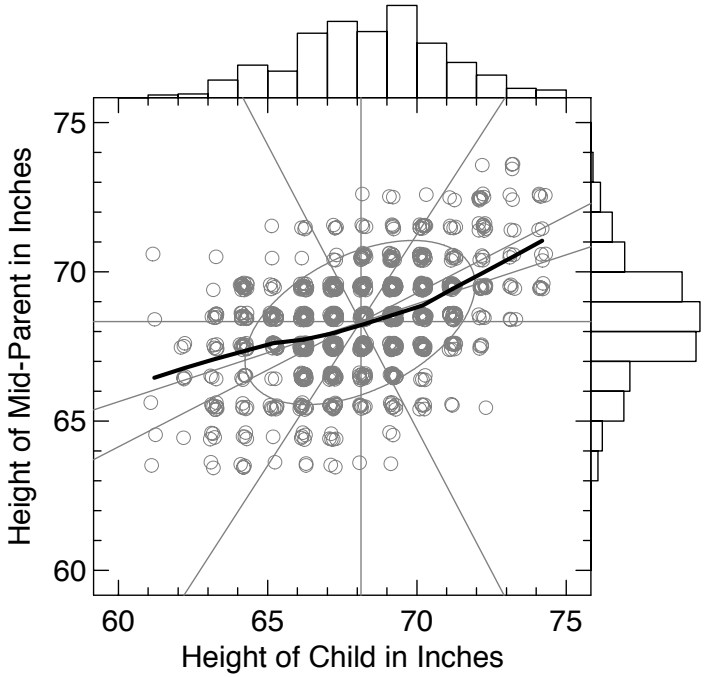
Predicting

Regression

Francis Galton (1887)

His data weren't as clean and linear as he imagined, but that didn't matter



Wachsmuth, Wilkinson & Dallal, 2003

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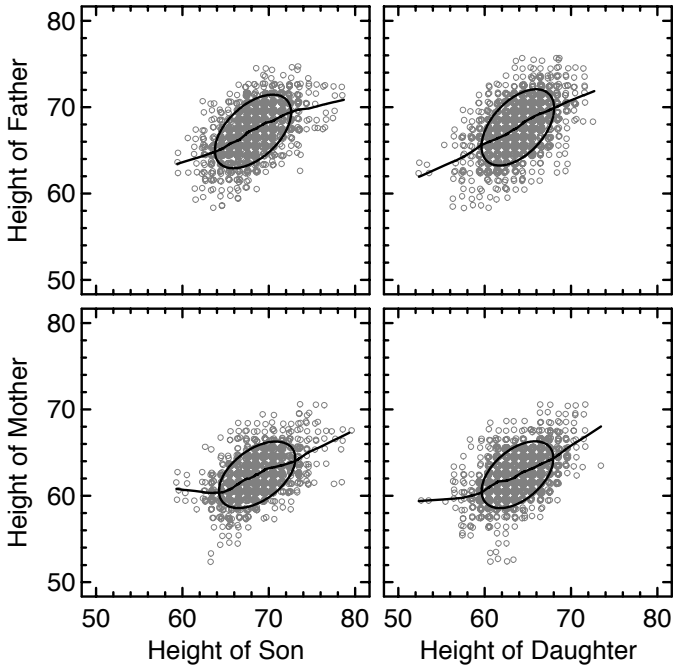
Predicting

Regression

Francis Galton (1887)

That's because he aggregated over different sources



Wachsmuth, Wilkinson & Dallal, 2003

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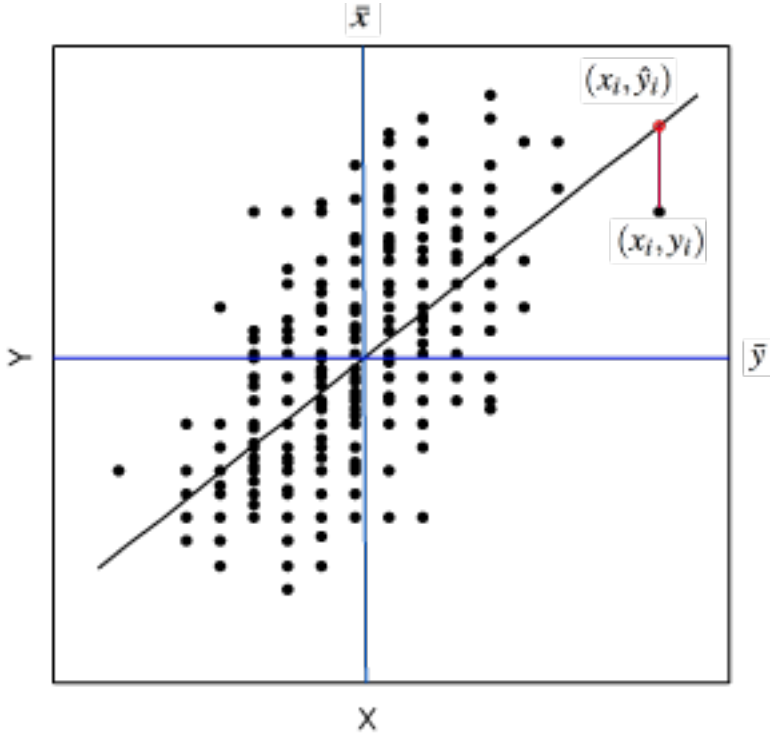
Predicting

Estimating via Ordinary Least Squares



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Predicting

Estimating via Ordinary Least Squares

For intercept (b_0) and slope (b_1), we could use calculus

- The way we did when we used maximum likelihood to estimate mean and sd
- In this case, we want to minimize the sum of squared residuals SSE
  - Given

$$y_i = b_0 - b_1 x_i + e_i$$

  - We sum the e_i to get SSE

$$SSE = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

- Compute the partial derivatives with respect to b_0 and b_1
- Set these derivatives to zero (where the minimum SSE exists)
- Solve the resulting simultaneous equations
- This is what Legendre originally did

But there is an easier way

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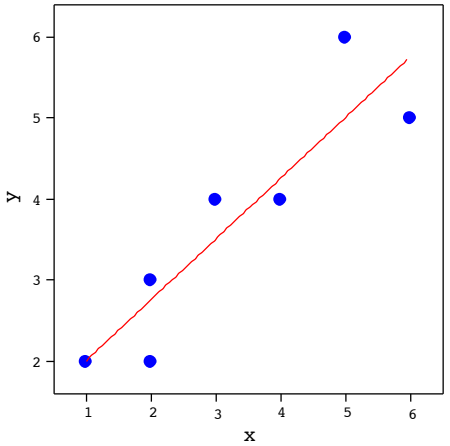
Predicting

Estimating via OLS (we'll use matrices)

$$
\begin{aligned}
Y &= XB + E \\
XB &\perp E \\
X'Y &= X'XB + X'E \\
(X'X)^{-1}X'Y &= (X'X)^{-1}(X'X)B \\
(X'X)^{-1}X'Y &= B \\
E &= Y - XB
\end{aligned}
$$



$$
Y = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \\ 4 \\ 6 \\ 5 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1.25 \\ 0.75 \end{bmatrix} \quad E = \begin{bmatrix} 0.25 \\ 0.00 \\ -.75 \\ 0.50 \\ -.25 \\ 1.00 \\ -.75 \end{bmatrix}
$$

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Predicting

Estimating the regression model parameters

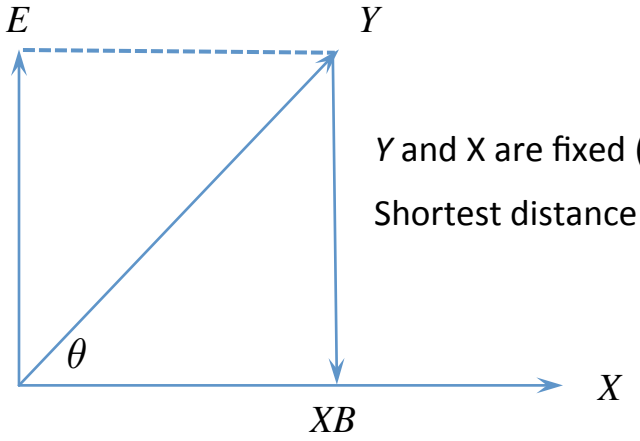
How do we know we minimized the error sum of squares?

$$X \cdot Y = \|X\|\|Y\| \cos\theta$$

$$\cos\theta = \frac{X \cdot Y}{\|X\|\|Y\|}$$

Pearson correlation coefficient (if data are centered)

$$\text{length of E} = \|E\|$$



Y and X are fixed (because of θ and $\|X\|$ and $\|Y\|$)

Shortest distance from point *Y* to line *X* is $\|E\|$ QED

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Predicting

Estimation (assuming we sampled from a population)

$$B = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

is a least-squares estimator

- unbiased (assuming residuals are homogeneous)
- smallest variance among unbiased estimators
- Best Linear Unbiased Estimator (BLUE)

If the residuals are normally distributed

- *B* is a maximum likelihood estimator
- We can do classical statistical tests on estimates of parameters

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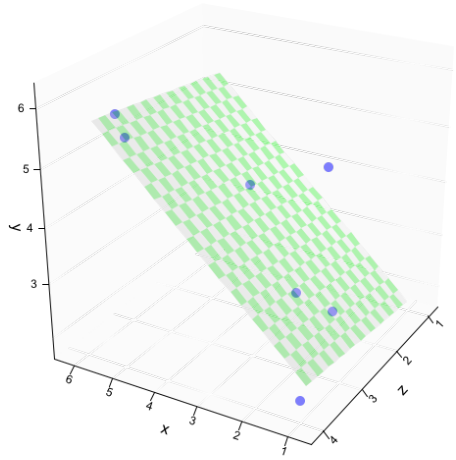
Predicting

Estimating via OLS

Two predictors (same formulas)

$$
\begin{aligned}
Y &= XB + E \\
XB &\perp E \\
X'Y &= X'XB + X'E \\
(X'X)^{-1}X'Y &= (X'X)^{-1}(X'X)B \\
(X'X)^{-1}X'Y &= B \\
E &= Y - XB
\end{aligned}
$$



$$
Y = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \\ 4 \\ 6 \\ 5 \end{bmatrix}
\quad
X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 4 & 2 \\ 1 & 5 & 4 \\ 1 & 6 & 3 \end{bmatrix}
\quad
B = \begin{bmatrix} 0.897 \\ 0.746 \\ 0.135 \end{bmatrix}
\quad
E = \begin{bmatrix} 0.206 \\ -.183 \\ -.659 \\ 0.730 \\ -.151 \\ 0.833 \\ -.778 \end{bmatrix}
$$

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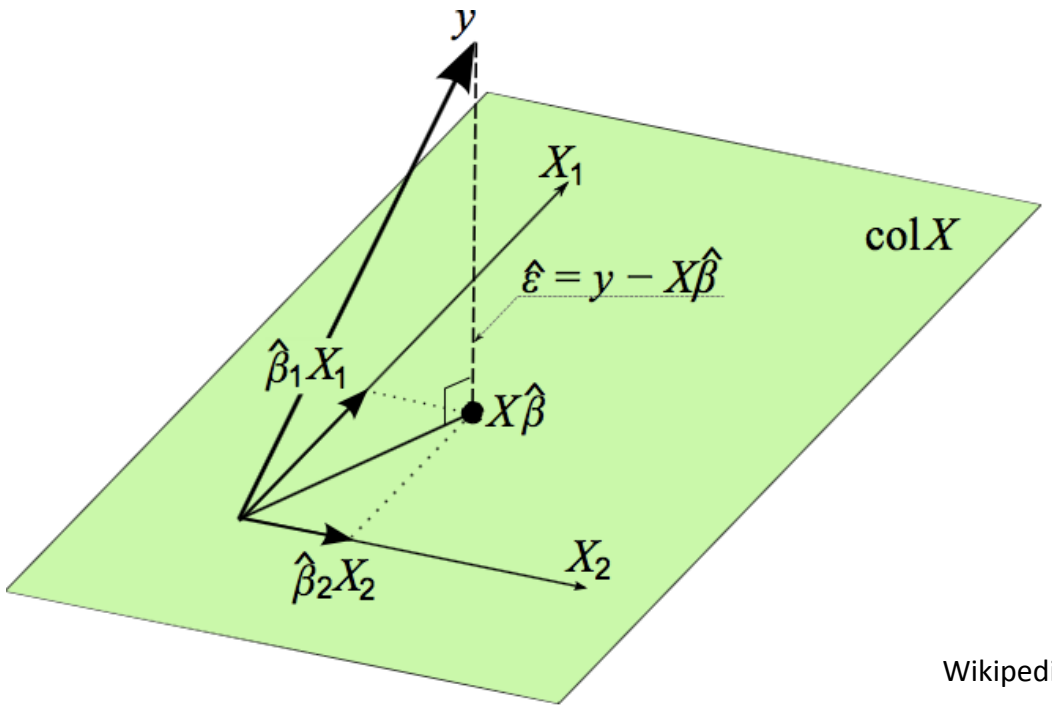
Predicting

The two-predictor vector space

$$Y = XB + E$$



Wikipedia

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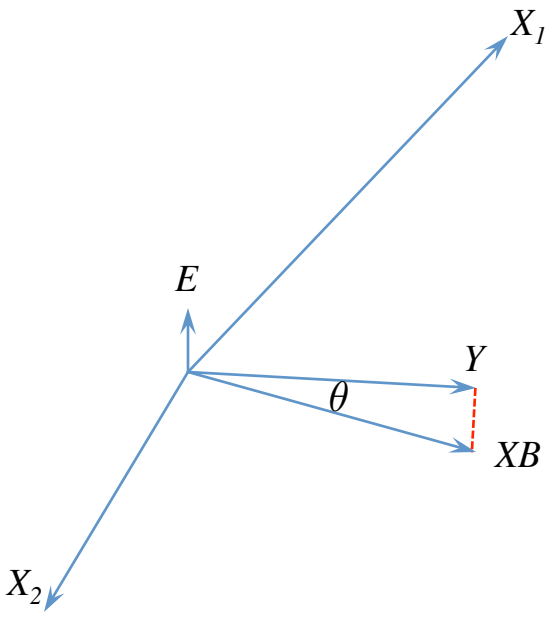
Predicting

The two-predictor vector space

It is possible for *y* not to be correlated much with either X_1 or X_2 yet be highly correlated with the linear combination of X_1 and X_2

So don't throw out predictors by looking at their correlations with the dependent variable



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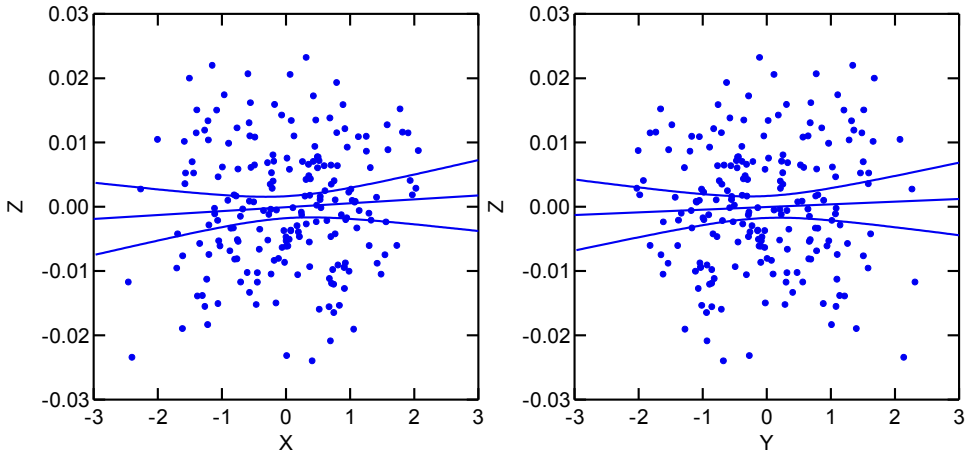
Predicting

The two-predictor vector space

Z is not significantly related to X

Z is not significantly related to Y

But the multiple correlation of Z with X and Y is almost 1!



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Predicting

Standardized Regression Coefficients (*Beta* Weights)

- Standardize variables, then compute regression
- Removes scales from consideration of size of coefficients
- Social scientists love this stuff

"Why then are correlation coefficients so attractive? Only bad reasons seem to come to mind. Worst of all, probably, is the absence of any need to think about units for either variable. Given two perfectly meaningless variables, one is reminded of their meaninglessness when a regression coefficient is given, since one wonders how to interpret its value."

Tukey (1969)

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Predicting

Sums of Squares

$$SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y}_i)^2$$

regression sum of squares (explained)

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

error sum of squares (unexplained)

$$SST = \sum_{i=1}^{n} (y_i - \bar{y}_i)^2$$

total sum of squares

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Predicting

Goodness of fit

Pearson correlation

$$\rho_{X,Y} = \frac{COV(X,Y)}{\sqrt{VAR(X)VAR(Y)}}$$

$$\hat{\rho}_{X,Y} = r_{X,Y} = \frac{X \cdot Y}{\|X\|\|Y\|}$$ (if *X* and *Y* are centered)

Multiple correlation (sqrt of coefficient of determination)

$$R^2 = \frac{\sum_{i=1}^{n}(\hat{y}_i - \bar{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y}_i)^2}$$ regression sum of squares / total sum of squares

$$R_{xx} = \begin{bmatrix} 1 & r_{x_1x_2} & \cdots & r_{x_1x_p} \\ r_{x_2x_1} & 1 & \cdots & r_{x_2x_p} \\ \vdots & \vdots & \vdots & \vdots \\ r_{x_px_1} & r_{x_px_2} & \cdots & 1 \end{bmatrix} \quad R_{yx} = \begin{bmatrix} r_{yx_1} & r_{yx_2} & \cdots & r_{yx_p} \end{bmatrix}$$

$$R^2 = R_{yx}R_{xx}^{-1}R_{xy}$$ $\approx$ sum of squared correlations with *Y* / correlations among *X*

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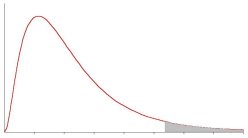
Predicting

Inference

F-statistic (test of significance for overall prediction)

$$F_{p,n-p-1} = \frac{R^2/p}{(1-R^2)/(n-p-1)}$$



Confidence intervals on regression coefficients

$$c_j = diag(X'X)_j^{-1}$$

$$s = \sqrt{\frac{SSE}{n-p-1}}$$

$$CI = (\hat{\beta} \pm t_{n-p-1}^{\alpha/2} s\sqrt{c_j})$$

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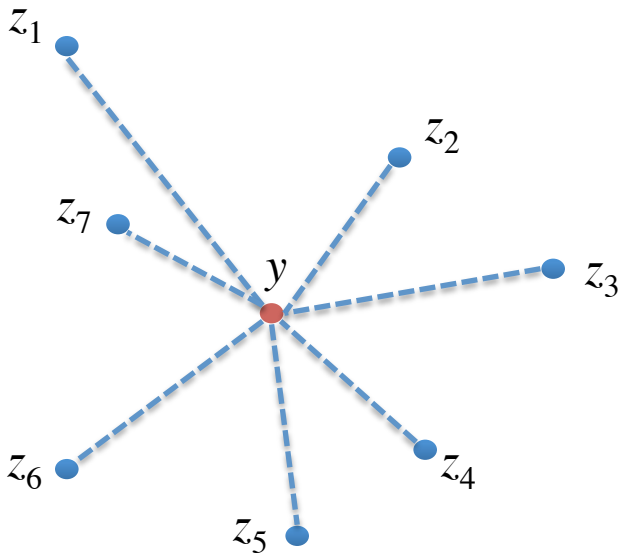
Predicting

An interesting application of Ordinary Least Squares

We have

- $\mathbf{Z}_{n \times p}$: *n* fixed points in *p* dimensions (cell towers, MDS or other configuration, …)
- $\mathbf{d}_n$: a new point $\mathbf{y}$'s estimated distance to each point in $\mathbf{Z}$

Solve for vector $\mathbf{b}_p$ specifying location of new point



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Predicting

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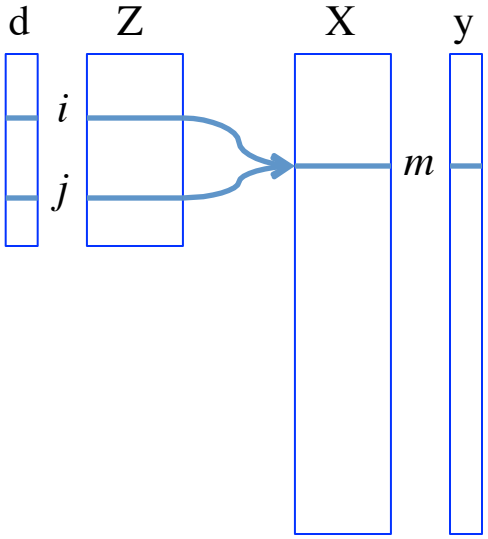

$$s = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \sum_{k=1}^{p} (\mathbf{Z}_{ik} - \mathbf{Z}_{jk})^2$$ (a scaling constant)

$$\mathbf{y}_m = \mathbf{d}_i^2 - \mathbf{d}_j^2 + s, \quad j = 1, \cdots, i-1, \; i = 2, \cdots, n, \; m = (i-1)(i-2)/2 + j$$

$$\mathbf{X}_{mk} = 2(\mathbf{Z}_{ik} - \mathbf{Z}_{jk}), \quad j = 1, \cdots, i-1, \; i = 2, \cdots, n, \; m = (i-1)(i-2)/2 + j$$

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

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Predicting

- OLS Assumptions
  - The true model is linear in the parameters
  - The *X* variables are not random and are measured without error
  - The *X* variables are not collinear
  - Residuals are uncorrelated/independent
  - The residuals have constant variance (homoscedasticity)
  - If *t* and *F* test statistics are computed
    - Residuals must be Normally distributed
    - Random sample from a population

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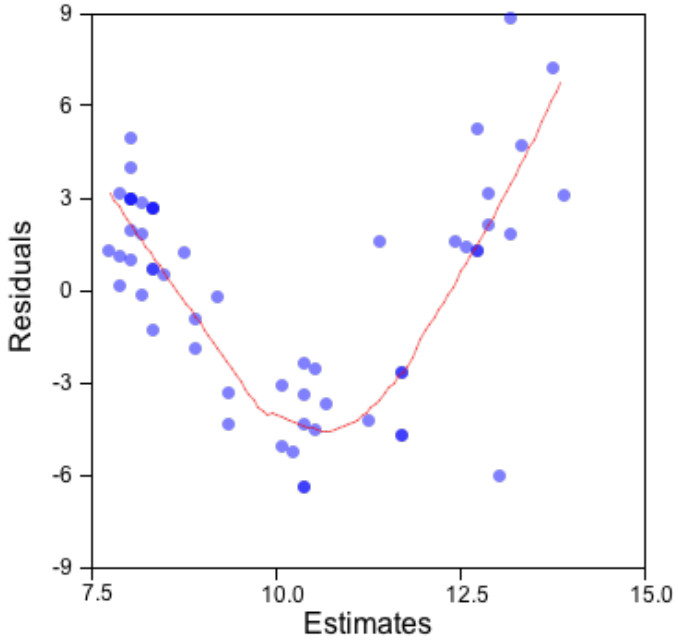
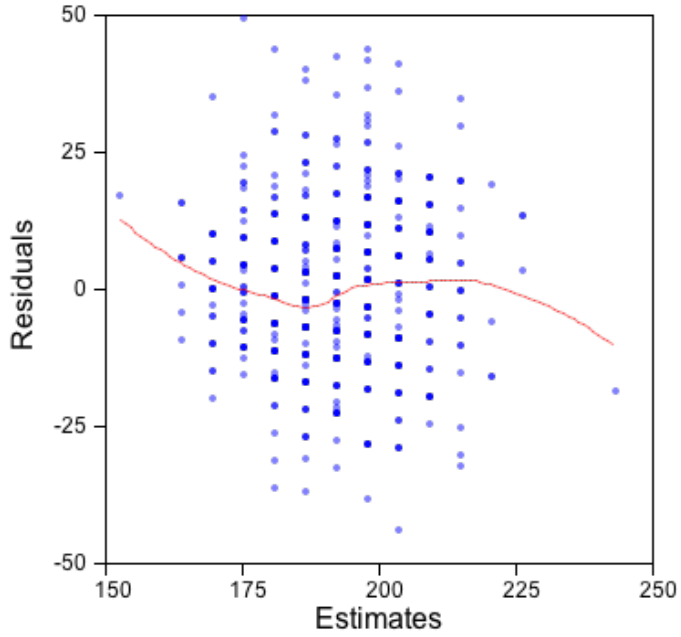
Predicting

Evaluating OLS assumptions

The true model is linear in the parameters

Use LOESS to put a smooth through residual plot





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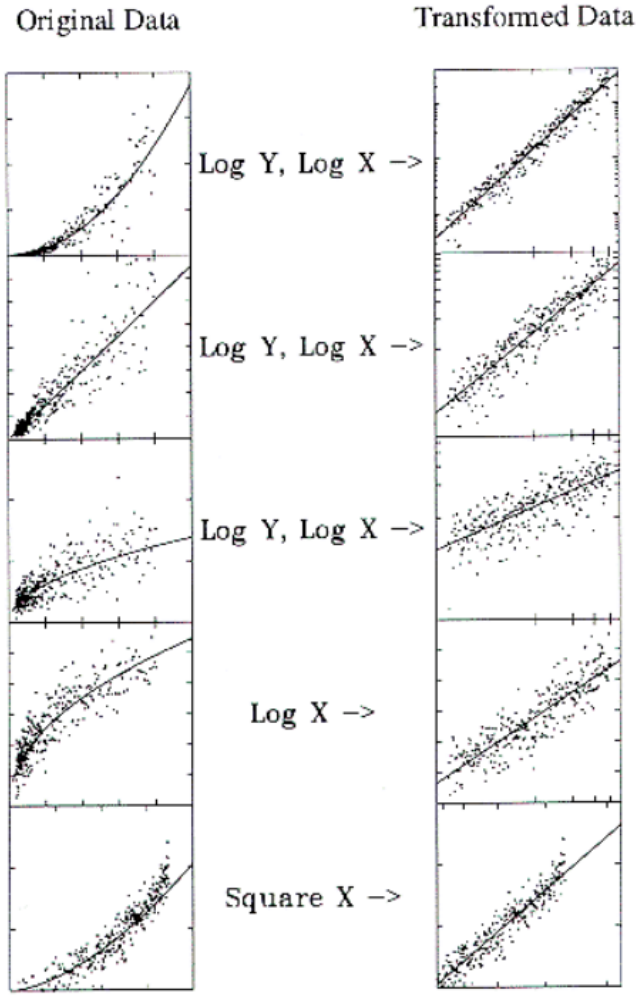
Predicting

Transformations

Dealing with nonlinearity



Wilkinson, Blank, & Gruber (1996)

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Predicting

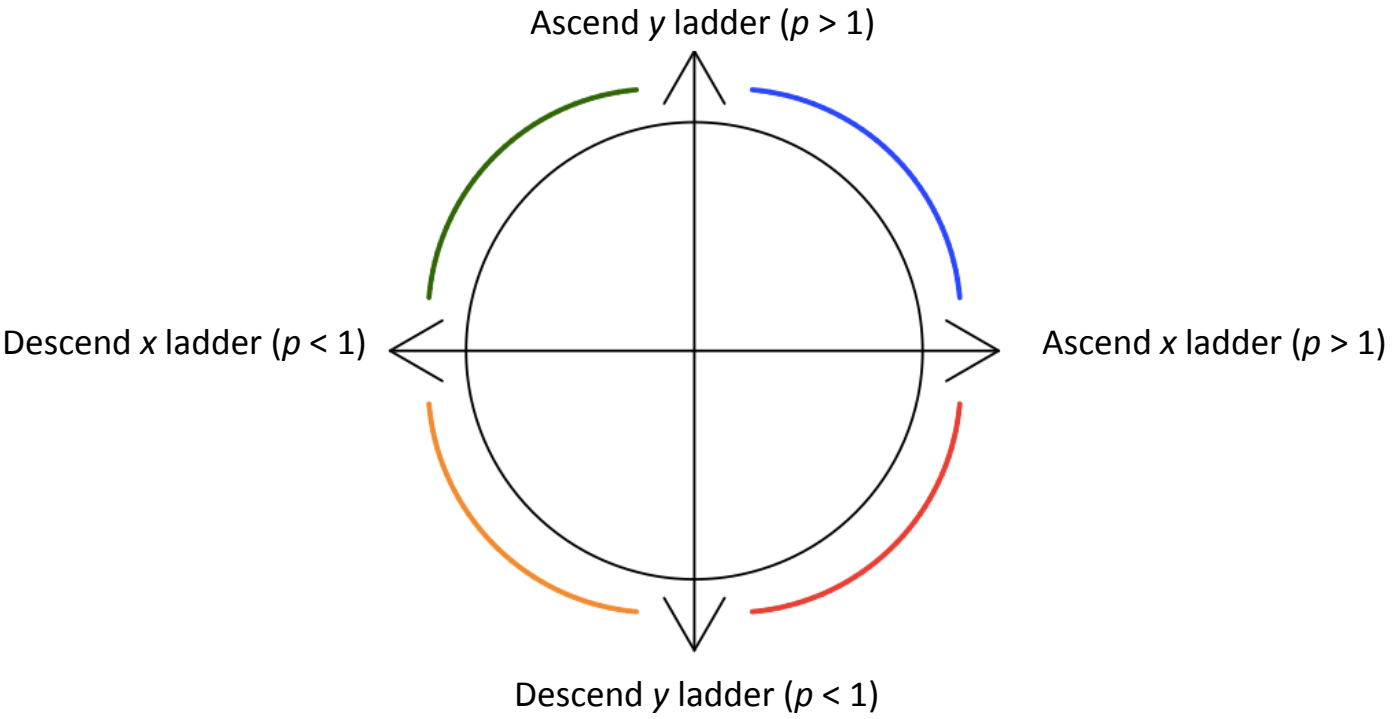
Transformations

Dealing with nonlinearity

Tukey-Mosteller bulging rule

$$x^* = x^p$$

$$y^* = y^p$$



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Predicting

Evaluating OLS assumptions

The *X* variables are not random and are measured without error

- There is no simple test or graphic for this
- Know the source of your measurements
- If there is measurement error, you can use errors-in-the-variables methods
- Or, you can just forgeddaboutit, which is what most of the world does
  - Who cares if your coefficient estimates are biased?
    - They are usually biased downward, so no harm done
    - Just don't show them to a social scientist
    - Social scientists love latent variable models
    - They are Platonists

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Predicting

Evaluating OLS assumptions

The *X* variables are not collinear

Values of Variance Inflation Factor (VIF) above 10 are worrisome $VIF = \frac{1}{1 - R_i^2}$

Figure 8. Variance Inflation Factors.

R^2 = .995

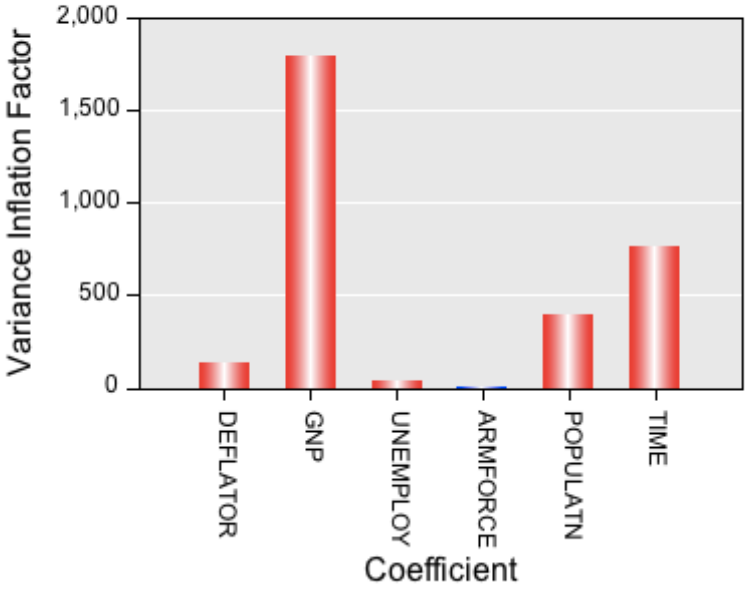
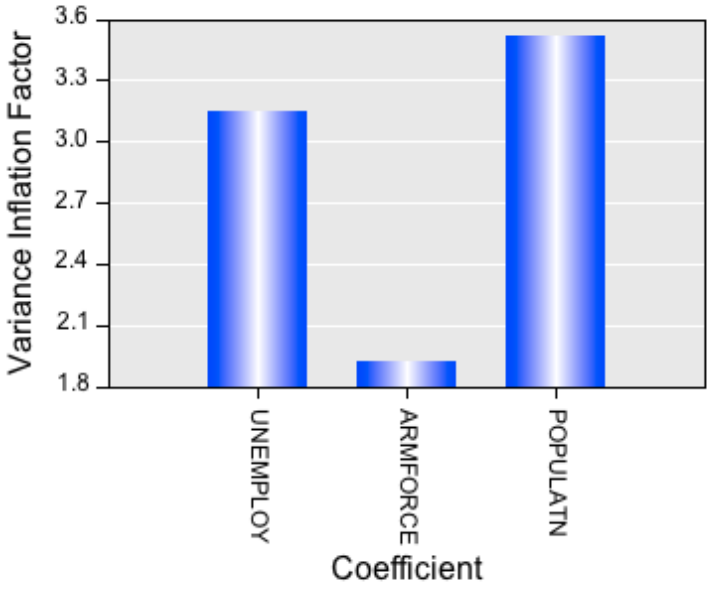


Figure 7. Variance Inflation Factors.

R^2 = .97



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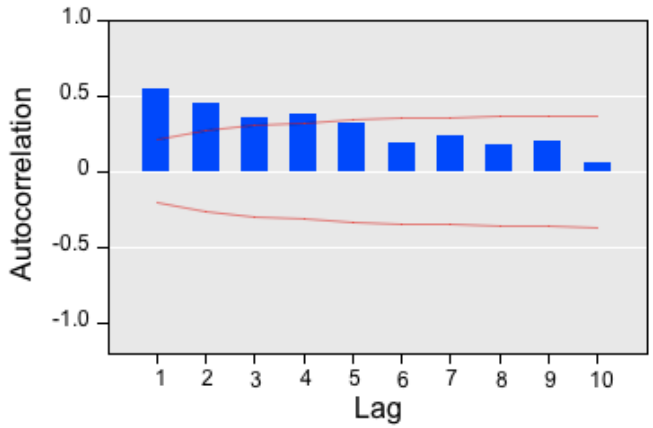
Predicting

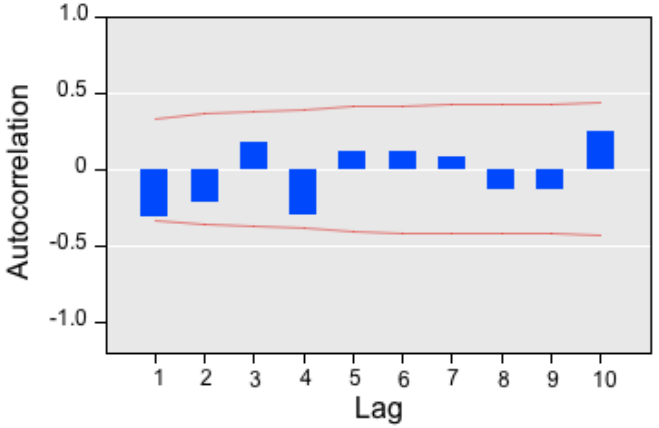
Evaluating OLS assumptions

Residuals are uncorrelated/independent

An ACF plot can help spot violations, but there are other types of dependencies

- Economists spend all their time worrying about this
- And for good reason – serial dependence is more toxic than outliers
- Don't even THINK of using ordinary linear regression (trend line) on time series data





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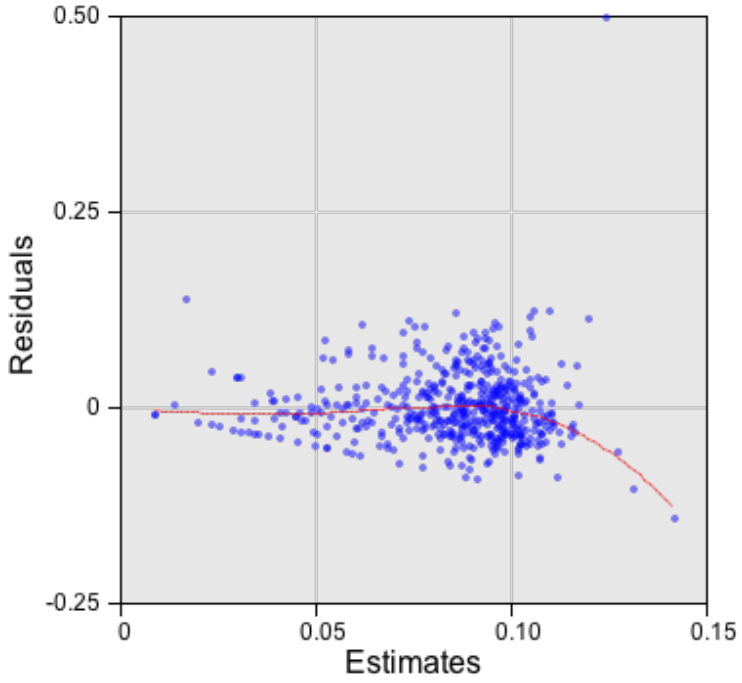
Predicting

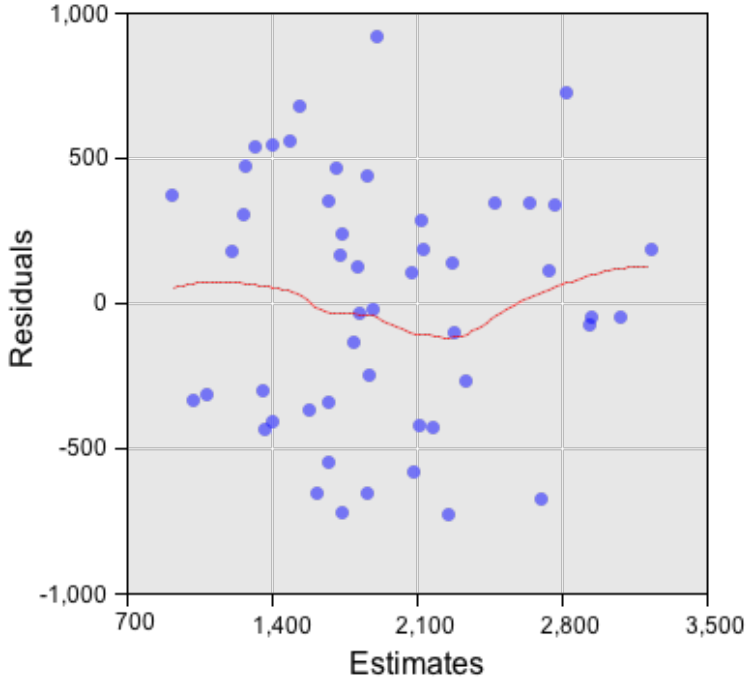
Evaluating OLS assumptions

The residuals have constant variance (homoscedasticity)

Try transformations (usually logging works for a power model)

Or use one of the heteroscedasticity corrections (MacKinnon-White, etc.)





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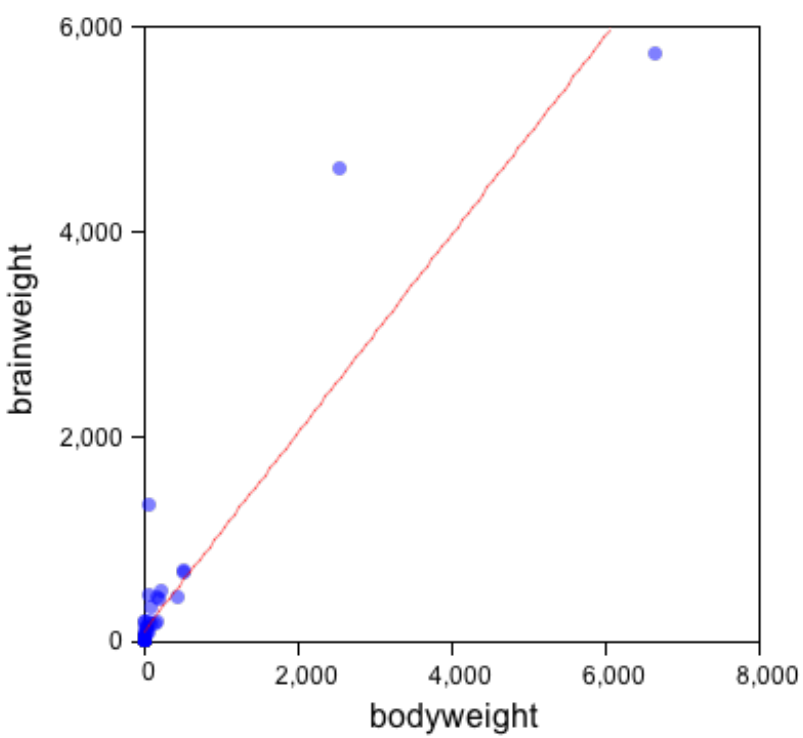
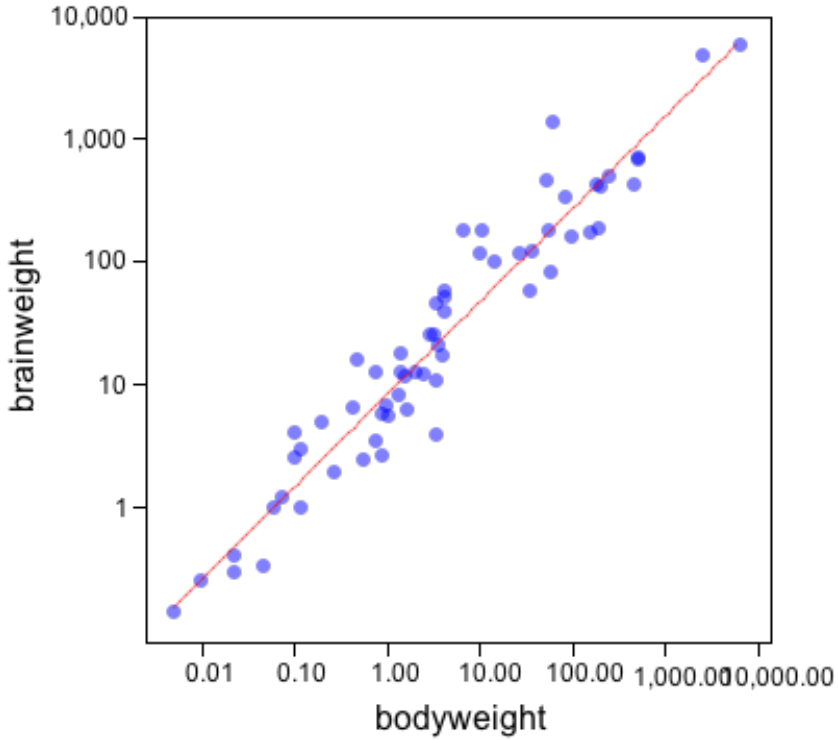
Predicting

Transformations

Dealing with heteroscedasticity

log-log transformation





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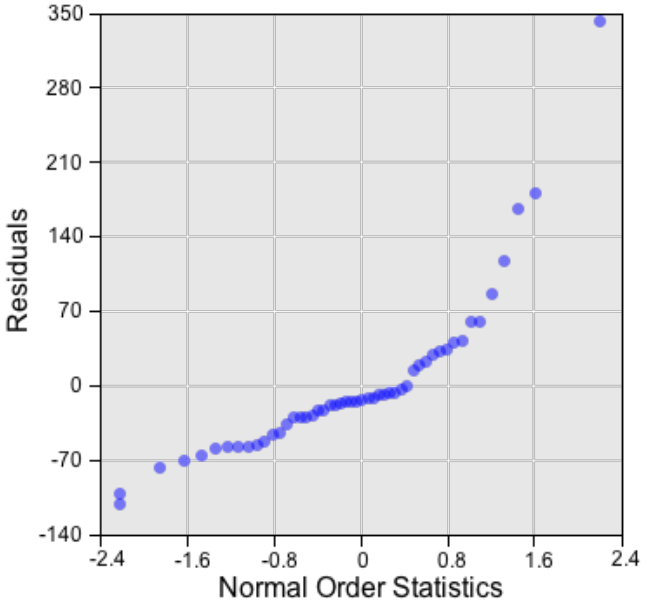
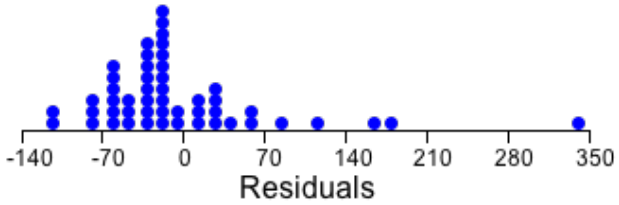
Predicting

Evaluating OLS assumptions

If *t* and *F* test statistics are computed

- Residuals must be Normally distributed
- Random sample from a population
- Forget about tests for normality; they are worthless





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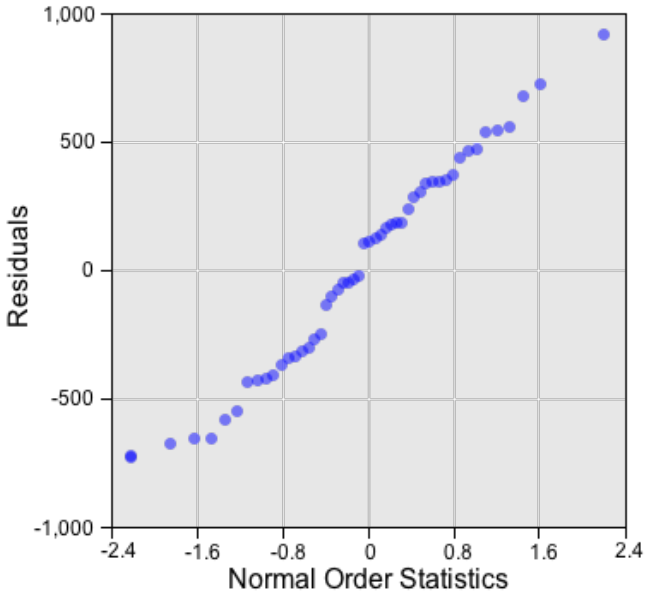
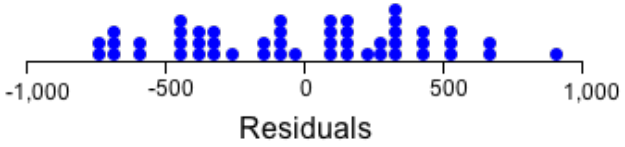
Predicting

Evaluating OLS assumptions

If *t* and *F* test statistics are computed

Residuals must be Normally distributed





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Predicting

Evaluating OLS assumptions

Leverage

Diagonal of the hat matrix (puts the hat on the parameters)

$$H = X(X'X)^{-1}X'$$

Cook's *D*

Measures the decrement in prediction by removing an observation

A type of leverage measure that is transformable to an approximate *F*

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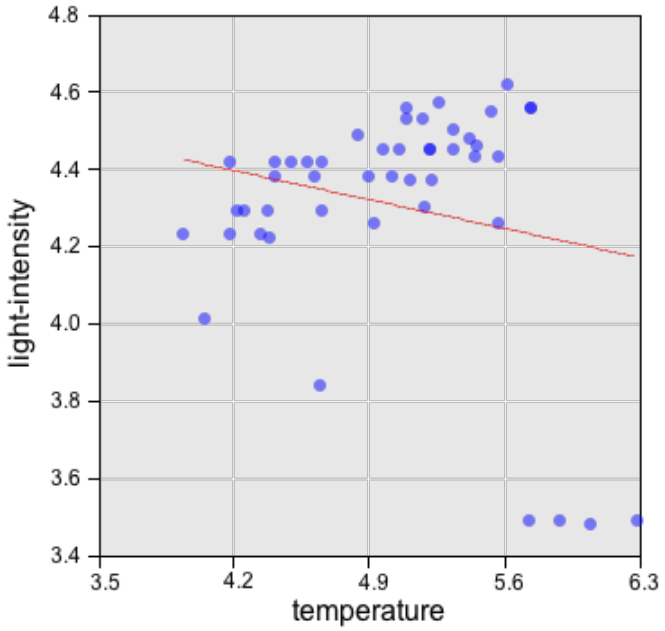
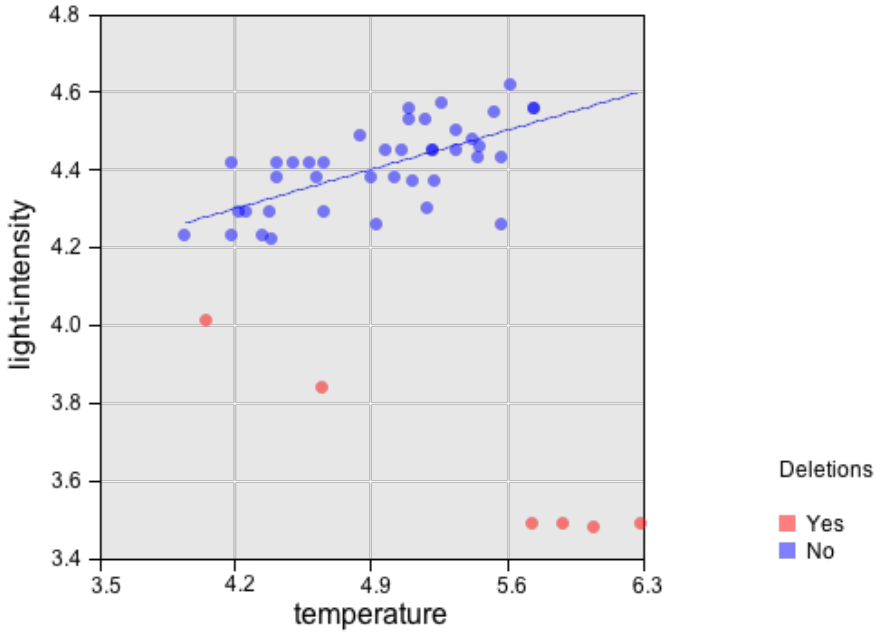
Predicting

Evaluating OLS assumptions

Leverage





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Predicting

Model Selection

Partial Residual Plots

Each plot consists of sets of residuals plotted against each other.

One set, on the vertical axis, consists of the residuals from regressing y on all the *X* (predictor) variables except for the predictor on the horizontal axis.

The other set, on the horizontal axis, consists of the residuals from regressing X_i on all the other *X* variables.

The result is a plot which shows you how each *X* variable is related to the *Y* variable when all the other *X* variables are taken into account.

Partial residual plots have several useful features:

- The slope of the line in the plot is the partial regression regression coefficient corresponding to the predictor in the plot.
- A line with a steep slope and nice looking residuals is a sign that a predictor belongs in the model.
- The residuals from this line are the same as the residuals from regressing *Y* on all the predictors.
- The plot helps you to judge whether the conditional relationship is linear or nonlinear.
- Extreme values on the horizontal axis help you to identify high leverage points.

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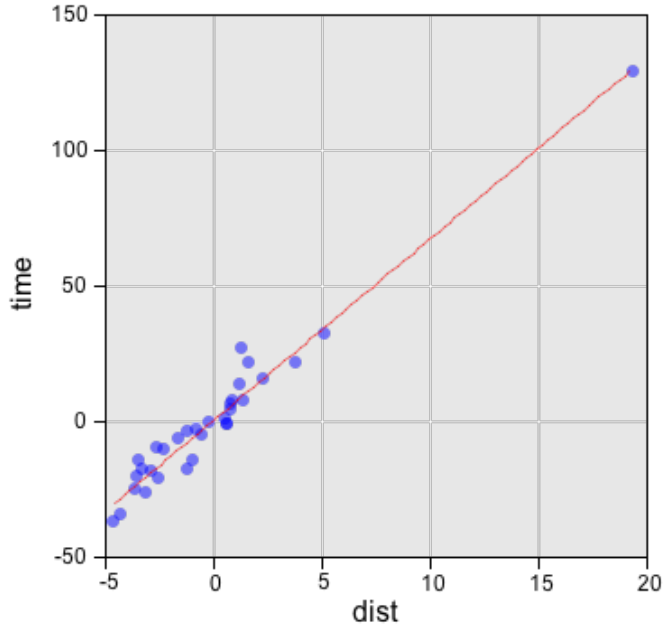
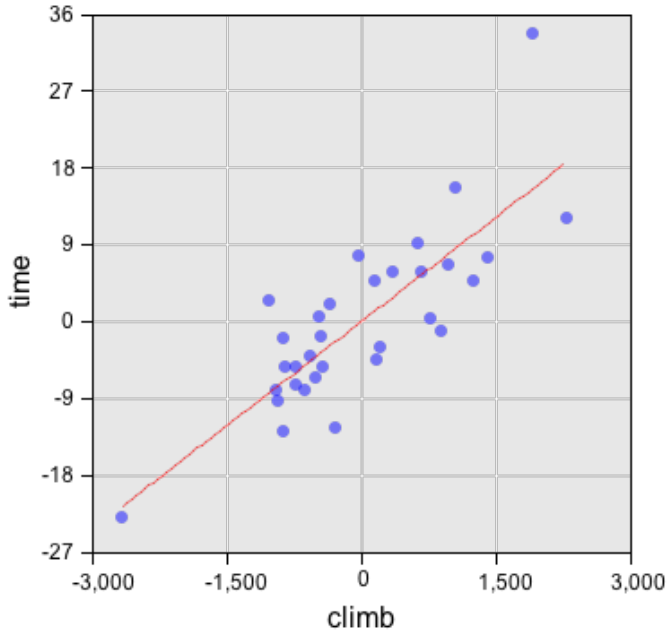
Predicting

Model Selection

Partial Residual Plots

$$\text{time} = -10.679 + 6.697 \times \text{distance} + 0.008 \times \text{climb}$$





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Predicting

- Model Selection
  - Forward Selection Regression
  - Backward Elimination Regression
  - Stepwise Regression
  - All Possible Subsets Regression

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Predicting

Model Selection

Wilkinson (1979) has discussed the case in which a subset of k predictors is to be chosen, where $1 < k < m$, and has provided tables of the upper 95th and 99th percentage points of the sample R^2 distribution in forward selection based on simulations (other tables and discussions of this problem can be found in Hocking, 1983; Rencher & Pun, 1980; and Wilkinson & Dallal, 1982). These tables are more conservative than the usual F tables. For example, with $N = 35$ and $\alpha = .05$, if all four members of a set of predictor variables are to be included in the regression equation, it is appropriate to use the standard F test to test R^2 for significance. When this is done, it is found that the sample R^2 has to exceed .26 in order to reject the hypothesis that the population multiple correlation coefficient is 0. However, if the four predictors are to be selected from a larger set of 20 predictors by a forward selection procedure, according to Wilkinson's tables, the sample R^2 must exceed .51 in order to reject the null hypothesis. Many researchers do not seem to be aware of this problem; for a sample of 66 published papers that reported significant forward selection analyses according to the usual F tests, Wilkinson found that 19 were not significant when his tables were used.

Myers, J.L., Well, A.D., Lorch Jr, R.F. (2013). *Research Design and Statistical Analysis* (3rd Ed.). Routledge.

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Predicting

Model Selection

Akaike Information Criterion (AIC)

Longley Data: Predicting TOTAL (RSQ = .995)

Constant	-3,482,258.635	890,420.384	-5,496,529.488	-1,467,987.781
DEFLATOR	15.062	84.915	-177.029	207.153
GNP	-0.036	0.033	-0.112	0.040
UNEMPLOY	-2.020	0.488	-3.125	-0.915
ARMFORCE	-1.033	0.214	-1.518	-0.549
POPULATN	-0.051	0.226	-0.563	0.460
TIME	1,829.151	455.478	798.788	2,859.515

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Predicting

Model Selection (*m* submodels)

$$AIC = 2(p+2) - 2\log(L)$$

$$\Delta AIC_k = AIC_k - AIC_{min} \quad \text{relative AIC}$$

$$L_k = \exp(-\Delta AIC/2) \quad \text{relative likelihood}$$

$$W_k = L_k / \sum_{i=1}^{m} L_i \quad \text{AIC weights}$$

	RSQ	Increment	AIC Weight
GNP&UNEMPLOY&ARMFORCE&TIME	0.995	0.051	0.503
UNEMPLOY&ARMFORCE&TIME	0.993	0.013	0.192
UNEMPLOY&ARMFORCE&POPULATN&TIME	0.995	0.027	0.167
GNP&UNEMPLOY&ARMFORCE&POPULATN&TIME	0.995	0.053	0.050
DEFLATOR&GNP&UNEMPLOY&ARMFORCE&TIME	0.995	0.073	0.049
DEFLATOR&UNEMPLOY&ARMFORCE&POPULATN&TIME	0.995	0.028	0.020
DEFLATOR&UNEMPLOY&ARMFORCE&TIME	0.993	0.016	0.016
DEFLATOR&GNP&UNEMPLOY&ARMFORCE&POPULATN&TIME	0.995	0.995	0.002
UNEMPLOY&TIME	0.982	0.009	0.001
GNP&UNEMPLOY&ARMFORCE	0.985	0.048	0.001
GNP&UNEMPLOY	0.981	0.047	0.000
GNP&ARMFORCE&POPULATN	0.984	0.013	0.000
DEFLATOR&UNEMPLOY&TIME	0.983	0.012	0.000

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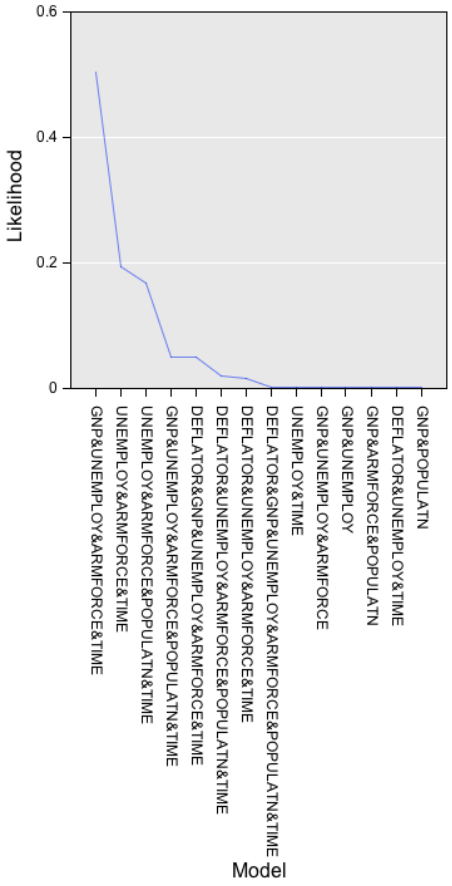
Predicting

Model Selection (*m* submodels)



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Predicting

Other approaches to multicollinearity

Regularization

Ridge regression (Hoerl, 1962)

Minimize

$$\sum_{i=1}^{n}(y_i - \mathbf{x}_i\boldsymbol{\beta})^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

So, $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}$

We inflate the diagonal to reduce the influence of the off-diagonal covariances

Ridge regression shrinks the estimates toward zero, introducing bias

But this reduces the variance of the estimates

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Predicting

Other approaches to multicollinearity

Regularization

LASSO (Least Absolute Value Shrinkage and Selection Operator)

Minimize

$$\sum_{i=1}^{n}(y_i - \mathbf{x}_i\boldsymbol{\beta})^2 + \lambda \sum_{j=1}^{p} |\beta_j|$$

Solution requires iteration

- Allows some coefficients to go to zero with others nonzero
- Efron and Tibshirani use Least Angle Regression with Shrinkage (LARS)
  - Similar to stepwise regression

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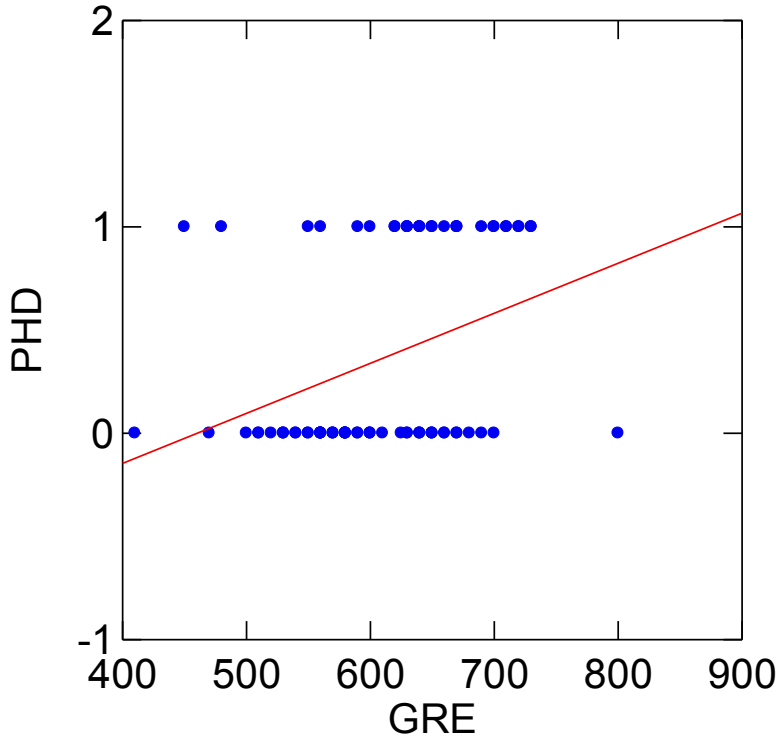
Predicting

Logistic Regression

OLS estimates



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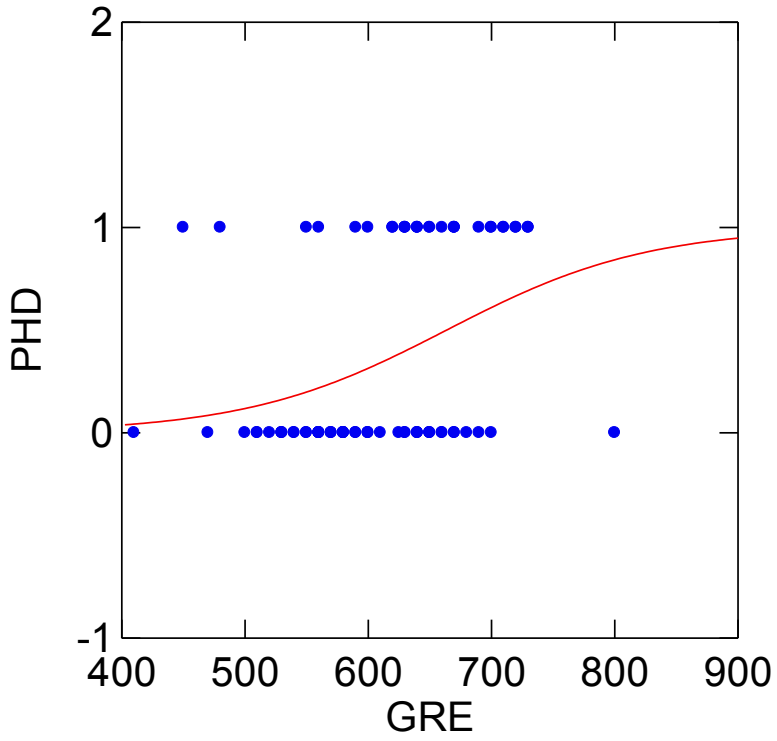
Predicting

Logistic Regression

LR estimates



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Predicting

Logistic Regression

Model and estimation

$$p(\mathbf{x};\boldsymbol{\beta}) = \frac{e^{\mathbf{x}\boldsymbol{\beta}}}{1+e^{\mathbf{x}\boldsymbol{\beta}}} = \frac{1}{1+e^{\mathbf{x}\boldsymbol{\beta}}}$$

$$\mathbf{x} = x_0, \cdots, x_p$$
$$\boldsymbol{\beta} = \beta_0, \cdots, \beta_p$$

$$logit(p) = \log \frac{p(\mathbf{x})}{1-p(\mathbf{x})} = \mathbf{x}\boldsymbol{\beta}$$

log-odds

$$L(\boldsymbol{\beta};\mathbf{x}_1,\ldots,\mathbf{x}_n) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$$

likelihood (Binomial)

$$l(\boldsymbol{\beta};\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{i=1}^{n} logit_i - \log(1+p_i)$$

log-likelihood

Must use iterative optimization to find maximum

Different model for more than two categories

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Predicting

Poisson Regression (log-linear model)

Model and estimation

$$E[Y|\mathbf{x}] = e^{\mathbf{x}\boldsymbol{\beta}}$$

Poisson distribution has only 1 parameter

$$p(y; \mathbf{x}, \boldsymbol{\beta}) = \frac{e^{y\mathbf{x}\boldsymbol{\beta}} e^{-e^{\mathbf{x}\boldsymbol{\beta}}}}{y!}$$

y is integer valued

$$L(\boldsymbol{\beta}; y, \mathbf{x}_1, \ldots, \mathbf{x_n}) = \prod_{i=1}^{n} \frac{e^{y_i \mathbf{x}_i \boldsymbol{\beta}} e^{-e^{\mathbf{x}_i \boldsymbol{\beta}}}}{y_i!}$$

likelihood

$$l(\boldsymbol{\beta}; y, \mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} \{(y_i \mathbf{x}_i \boldsymbol{\beta}) - e^{\mathbf{x}_i \boldsymbol{\beta}} - \log(y_i!)\}$$

log-likelihood
(don't need the yellow term
In order to maximize)

Must use iterative optimization to find maximum

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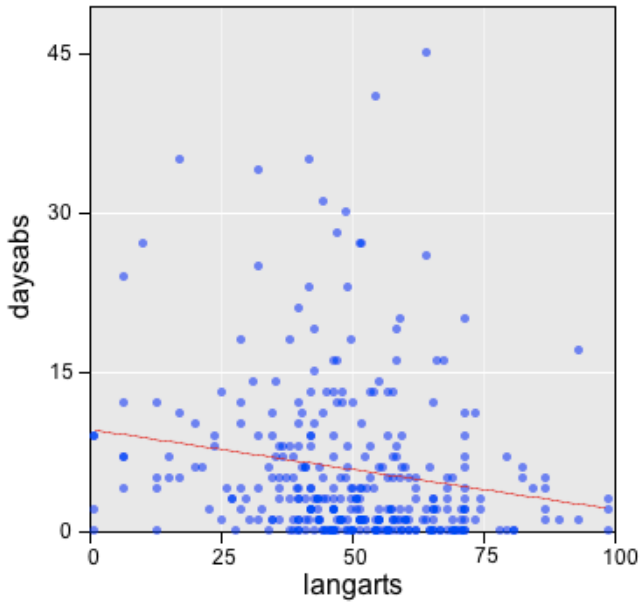
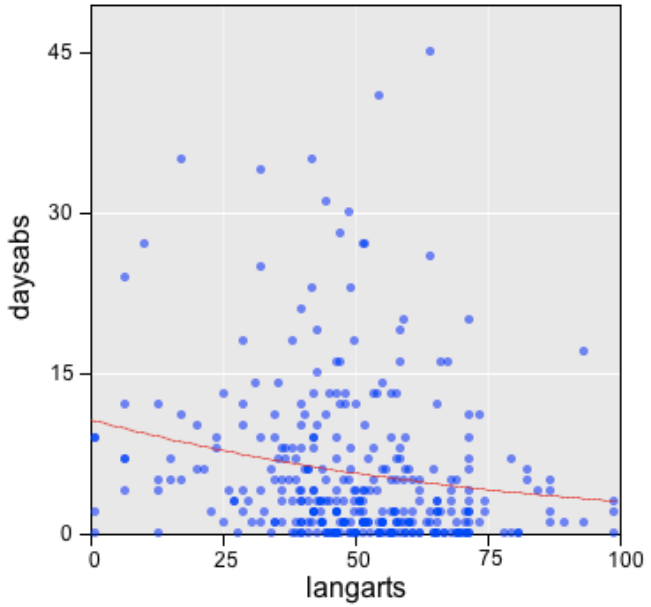
Predicting

Poisson Regression

OLS



Poisson



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Predicting

Generalized Linear Models (GLM)

Nelder & Wedderburn (1972)

Not to be confused with

- General Linear Model (GLM) for OLS with or without dummy variables
- Generalized Least Squares (GLS) for dealing with heteroscedasticity

Estimation done through Iteratively Reweighted Least Squares (IRWLS)

$\mathbf{x}\boldsymbol{\beta} = g(E[Y])$ link function

$E[Y] = \mu = g^{-1}(\mathbf{x}\boldsymbol{\beta})$ modeling mean of *Y* through inverse of link function

Distribution	Link Function Name	Link Function
Normal	Identity	$\mathbf{x}\boldsymbol{\beta} = \mu$
Binomial	Logit	$\mathbf{x}\boldsymbol{\beta} = \log\left(\frac{\mu}{1-\mu}\right)$
Poisson	Log	$\mathbf{x}\boldsymbol{\beta} = \log(\mu)$

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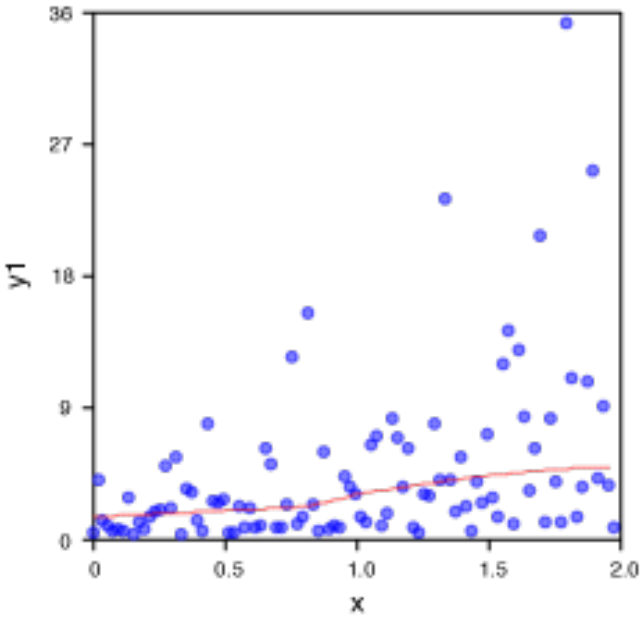
Predicting

Nonlinear Regression

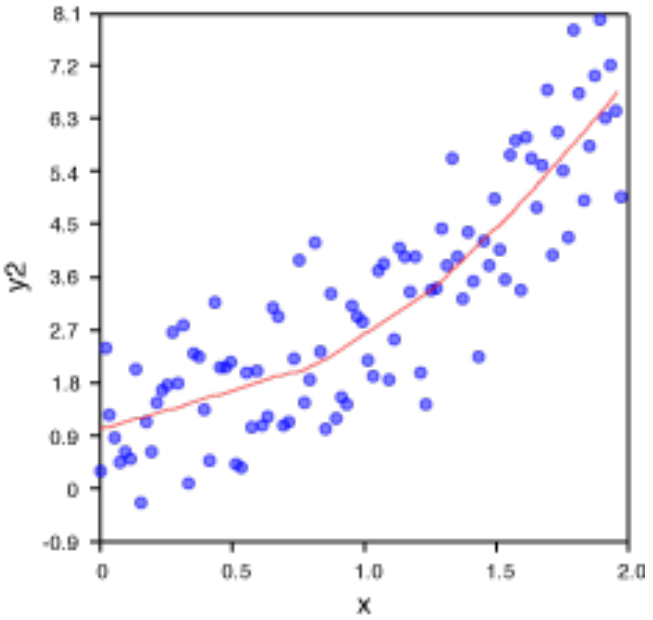
Functions of the form $y = f(x + \varepsilon)$

Left is linearizable by transformation, right is intrinsically nonlinear

$$y = e^{\beta_0+\beta_1 x+\epsilon}$$



$$y = e^{\beta_0+\beta_1 x} + \epsilon$$



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Predicting

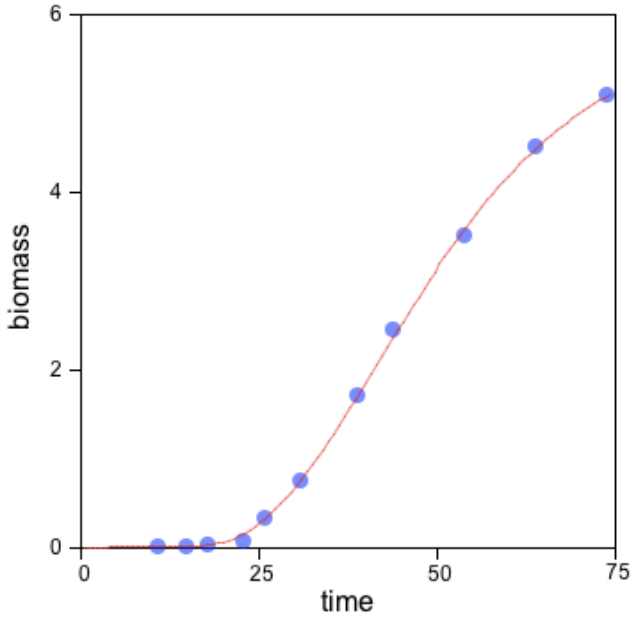
Nonlinear Regression

$$E[Y] = e^{a+b/x+c\log(x))}$$

Minimize SSE (Newton, Quasi-Newton, Metropolis, …)

Magic, right?

	Coefficient	Standard Error	Lower95%	Upper95%
a	12.547	1.134	10.324	14.769
b	-192.035	12.066	-215.684	-168.385
c	-1.934	0.227	-2.379	-1.490



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Predicting

Nonlinear Regression

Things to worry about

- Don't waste your time with R^2
  - There are all sorts of definitions
  - All are nonsense
  - The bad ones are ridiculously large
  - Look at sum of squared errors instead
- Don't hunt around for nonlinear equations that work
  - That's a fishing expedition
  - The equation you choose should be driven by theory and the domain it applies to
- Half the time your iterative fitting method will croak
  - That's a sign that there's a local minimum
  - Or your equation is wrong
  - Or your starting values are wildly off-target
  - Don't waste your time looking for another computer program
    - Nonlinear programs are notoriously finicky
  - Your data are probably crap or your model is ridiculous
- Look at the fit before you examine any statistics
  - Most of the time you have one predictor and one dependent variable
  - So look at the fitted equation
  - Ignore the fit statistics and tests of significance until it looks good
- EVERYTHING fits
  - Keep in mind that almost any bad model will look pretty good
  - If it doesn't make theoretical sense, don't trust it

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Predicting

References

McCullagh, P. and Nelder, J. (1989). *Generalized Linear Models, Second Edition*. Boca Raton: Chapman and Hall/CRC.

Tukey, J.W. (1969). Analyzing data: Sanctification or detective work? *American Psychologist, 24*, 83-91.

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